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Integral stability criteria of nonlinear differential systems

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Abstract

The notions of equi-integral and ϕ_0-stability for systems of ordinary differential equations (ODEs) were introduced and extended. The notions of equi-integral stability are extended to the so-called equi-integral ϕ_0-stability of perturbed system of (ODEs). Our technique depends on cone-valued Liapunov function method.

Keywords: Equi-integrally stable; Equi-integrally ϕ_0-stable; Equi-asymptotically integrally stable; Equi-asymptotically integrally ϕ_0-stable

1. Introduction

The problems of the qualitative properties of differential equations via Liapunov direct method has been successfully investigated in a unified way using the comparison principle [7]. In this method the qualitative properties of the system of differential equations are inferred from the corresponding properties of solutions of the comparison systems. Zhang [12] used this method to discuss boundedness of solutions. Lakshmikantham and Leela [7] used the comparison principle to discuss different notions of stability namely, eventual and integral stability. Moreover the same authors in [8] initiated the development theory of differential inequalities through cone and cone-valued Liapunov function method. In [5,6], the notion of ϕ_0-stability [2] for comparison systems, were extended to some new types of stability, using Liapunov direct method, cone-valued Liapunov function method [8] and comparison technique.

Our purpose in this paper is to extend ϕ_0-stability notion [2] to a new type of stability namely integral ϕ_0-stability. The notion of uniform integral ϕ_0-stability lies somewhere between uniform integral stability of [7] on one side and uniform ϕ_0-stability of [2] on the other side. The motivation for this work is the recent results of [2,3,5,6,11].

Let $\mathfrak{R}^n$ denote the n-dimensional Euclidean space with any convenient norm $\|.\|$, and the scalar product $(x, y) \leq \|x\|\|y\|$, $\mathfrak{R}^+ = [0, \infty)$, $J = [t_0, \infty)$, $\mathfrak{R} = (-\infty, \infty)$, $\mathfrak{R}^n_+ = \{x \in \mathfrak{R}^n, x_i \geq 0, i = 1, 2, \ldots, n\}$, and let $C[\mathfrak{R}^+ \times \mathfrak{R}^n, \mathfrak{R}^n]$ denote the space of continuous functions that map $\mathfrak{R}^+ \times \mathfrak{R}^n$ into $\mathfrak{R}^n$.

The following definitions will be needed in what follows.

Definition 1.1 (*[7]*). A function $b(r)$ is said to belong to the class $\mathcal{K}$ if $b(0) = 0, b \in C[[0, \rho), \mathfrak{R}^+]$, and $b(r)$ is strictly monotone increasing in r.

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Definition 1.2 (*[2]*). A proper subset K of $\mathfrak{R}^n$ is called a cone if

$$\text{(i) } \lambda K \subset K, \quad \lambda \geq 0, \qquad \text{(ii) } K + K \subset K, \qquad \text{(iii) } \overline{K} = K, \qquad \text{(iv) } K^\circ \neq \emptyset, \qquad \text{(v) } K \cap (-K) = \{0\},$$

where $\overline{K}$ and K° denote the closure and interior of K respectively, and ∂K denotes the boundary of K. The set

$$K^\star = \{\phi \in \mathfrak{R}^n ; (\phi, x) \geq 0, x \in K\}$$

is called the adjoint cone if it satisfies the properties (i)–(v) of Definition 1.2.

$x \in \partial K$ iff $(\phi, x) = 0$, for some $\phi \in K_0^\star$, $\quad K_0 = K \setminus \{0\}$.

Definition 1.3 (*[2]*). A function $L : D \to \mathfrak{R}^n$, $D \subset \mathfrak{R}^n$ is called quasimonotone relative to the cone K if $x, y \in D$ and $y - x \in \partial K$, then there exists $\phi_\circ \in K_\circ^\star$ such that $(\phi_\circ, y - x)$ and $(\phi_\circ, g(y) - g(x)) \geq 0$.

Consider the nonlinear system

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

and the perturbed system

$$x' = f(t, x) + h(t, x), \quad x(t_0) = x_0, \tag{1.2}$$

where $f, h \in C[\mathfrak{R}^+ \times S_\rho, \mathfrak{R}^n]$, $f(t, 0) = h(t, 0) = 0$. Following [7], we define a Liapunov function $V(t, x) \in C[\mathfrak{R}^+ \times S_\rho, K]$ for $(t, x) \in \mathfrak{R}^+ \times S_\rho$, and the upper right-hand derivative of $V(t, x)$ as follows

$$D^+ V(t, x) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}[V(t + \delta, x + \delta f(t, x)) - V(t, x)].$$

Consider the comparison differential system

$$u' = g(t, u), \quad u(t_0) = u_0, \tag{1.3}$$

and

$$u' = g(t, u) + p(t), \quad u(t_0) = u_0, \tag{1.4}$$

where $g(t, 0) = 0$, $g \in C[\mathfrak{R}^+ \times K, \mathfrak{R}^n]$ and $p(t) \in C[J, \mathfrak{R}^+]$, K is a cone in $\mathfrak{R}^n$.

Let $S(\rho) = \{u \in K : \|u\| < \rho, \rho > 0\}$, $V \in C[\mathfrak{R}^+ \times S(\rho), K]$, and define for $(t, u) \in \mathfrak{R}^+ \times S(\rho)$, $\delta > 0$, the function $D^+ V(t, u)$ by

$$D^+ V(t, u) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}[V(t + \delta, x + \delta g(t, u)) - V(t, u)].$$

Definition 1.4 (*[7]*). The zero solution of (1.1) is said to be equi-integrally stable, if for $\alpha \geq 0$, $t_0 \in J$, there exists a positive function $\beta(t_0, \alpha)$ which is continuous in t_0 for $\alpha, \beta \in \mathcal{K}$ such that for every solution $x(t, t_0, x_0)$ of the perturbed system (1.2), the inequality

$$\|x(t, t_0, x_0)\| < \beta, \quad t \geq t_0,$$

holds, provided that

$$\|x_0\| \leq \alpha \quad \text{and} \quad \int_{t_0}^{t_0 + T} \sup_{\|x\| < \beta} \|h(s, x(s))\| \mathrm{d}s \leq \alpha, \quad \text{for every } T > 0.$$

Any integral stability notion can be similarly defined (see [7]).

The following definitions are somewhat new and related with that of [2,3,5,7].

Definition 1.5. The zero solution of (1.1) is said to be equi-integrally ϕ_0-stable, if for $\alpha \geq 0$, $t_0 \in J$, there exists a positive function $\beta(t_0, \alpha)$ which is continuous in t_0, for $\alpha, \beta \in \mathcal{K}$ such that for $\phi_0 \in K_0^*$, the inequality

$$(\phi_0, x^*) < \beta$$

holds provided that

$$(\phi_0, x_0) \leq \alpha \quad \text{and} \quad \int_{t_0}^{t_0+T} \sup_{\|x\|<\beta} \|h(s, x^*)\| \mathrm{d}s \leq \alpha, \quad \text{for every } T > 0,$$

where x^* denotes the maximal solution of the perturbed system (1.2) satisfies $x^*(t_0) = x_0$. In the case of uniform integral ϕ_0-stability, β is independent of t_0.

Definition 1.6 (*[1]*). The zero solution of (1.1) is said to be equi-asymptotically integrally ϕ_0-stable, if Definition 1.5 holds, and for every $\epsilon > 0$, $\alpha \geq 0$, and $t_0 \in J$, there exist positive function $\beta(t_0, \alpha)$, positive numbers $T = T(t_0, \alpha, \epsilon)$ and $\gamma = \gamma(t_0, \alpha, \epsilon)$ such that for the maximal solution $x^*(t, t_0, x_0)$ of (1.2) and $\phi_0 \in K_0^*$, the inequality

$$(\phi_0, x^*) < \epsilon, \quad t \geq t_0 + T,$$

holds, provided that

$$(\phi_0, x_0) \leq \alpha \quad \text{and} \int_{t_0}^{t_0+T} \sup_{\|x\|\leq\beta} \|h(s, x^*)\| \mathrm{d}s < \gamma.$$

Following [7], the fundamental matrix solution $\Phi(t, t_0, x_0)$ of the differential equation

$$E' = f_x(t, x(t, t_0, x_0))E, \tag{1.5}$$

is given by

$$\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0}.$$

2. Integral stability

In this section, we extend the notion of ϕ_0-equistable mentioned in [1] to the notion of equi-integral ϕ_0-stability for the system (1.1).

Theorem 2.1. *Assume that there exist two functions $V(t, x) \in C[\mathfrak{R}^+ \times S_\rho, K]$ and $g(t, x) \in C[\mathfrak{R}^+ \times K, \mathfrak{R}^n]$, with $V(t, 0) = g(t, 0) = 0$, further assume that $f, h \in C[\mathfrak{R}^+ \times K, \mathfrak{R}^n]$, $f(t, x)$ and $h(t, x)$ are quasimonotone in x relative to K, satisfying*

- (H_1) *$(\phi_0, x) \leq (\phi_0, V(t, x))$,*
- (H_2) *$V(t, x)$ is Lipschitzian in x relative to K,*
- (H_3) *$D^+(\phi_0, V(t, x)) \leq g(t, V(t, x))$, and*

$$\int_{t_0}^{t} g(s, V(s, x(s)))\mathrm{d}s \leq V(t_0, x_0).$$

If the zero solution of (1.3) is equi-integrally ϕ_0-stable, then so is the zero solution of (1.1).

Proof. Let the zero solution of (1.3) be integrally ϕ_0-equistable, then for $\alpha > 0$, $t_0 \in J$, there exists a positive function $\beta(t_0, \alpha)$ which is continuous in t_0, for $\alpha, \beta \in \mathcal{K}$, such that for $\phi_0 \in K_0^*$,

$$(\phi_0, r^*) < \beta, \tag{2.1}$$

provided that

$$(\phi_0, u_0) \leq \alpha, \quad \text{and} \quad \int_{t_0}^{t_0+T} p(s)\mathrm{d}s \leq \alpha, \quad T > 0, \tag{2.2}$$

where r^* is the maximal solution of (1.4). Choose $h(t, x^*) = Mp(t)$, $M > 1$, and $u_0 = V(t_0, x_0)$, thus

$$\int_{t_0}^{t_0+T} h(s, x^*)\mathrm{d}s = M \int_{t_0}^{t_0+T} p(s)\mathrm{d}s < \alpha = \alpha^*.$$

By integrating the first inequality and using the second inequality in (H_3), we get

$$\begin{aligned} V(t, x^*(t, t_0, x_0)) &\le V(t_0, x_0) + \int_{t_0}^{t_0+T} g(s, V(s, x(s)))\mathrm{d}s \\ &\le V(t_0, x_0). \end{aligned} \tag{2.3}$$

Now, by using Theorem 1.4.1 of [7], it follows that

$$V(t, x^*(t, t_0, x_0)) \le r^*(t, t_0, u_0), \tag{2.4}$$

where $r^*(t, t_0, u_0)$ and $x^*(t, t_0, x_0)$ are the maximal solutions of (1.4) and (1.2) respectively, thus for $\phi_0 \in K_0^*$, the inequalities (2.3) and (2.4) become

$$(\phi_0, V(t, x^*(t, t_0, x_0))) \le (\phi_0, V(t_0, x_0)) \tag{2.5}$$

$$\le (\phi_0, r^*). \tag{2.6}$$

Thus from condition (H_1), and the inequalities (2.1) and (2.6) we get

$$(\phi_0, x^*) \le (\phi_0, V(t, x^*)) \le (\phi_0, r^*(t)) < \beta$$

provided that

$$(\phi_0, x_0) < \alpha^* \quad \text{and} \quad \int_{t_0}^{t_0+T} \sup_{\|x\|<\beta} \|h(s, x^*)\|\mathrm{d}s \le \alpha^*.$$

Then the zero solution of (1.1) is equi-integrally ϕ_0-stable. □

Remark. (1) It is easy to see that if the zero solution of (1.1) is equi-integrally stable, then it is integrally ϕ_0-stable.
(2) Condition (H_1) is more relaxed than condition (H_{10}) of [6].

Theorem 2.2. *Let the hypothesis* (H_2) *be satisfied and for some* $\phi_0 \in K_0^*$, $(t, x) \in J \times S_\rho$, *the following conditions be satisfied*

(H_4) $a(\phi_0, x) \le (\phi_0, V(t, x))$, $a^{-1}(\alpha) = \alpha$, $a \in \mathcal{K}$, $\alpha > 0$,
(H_5) $f, h \in C[\Re^+ \times K, \Re^n]$, $f(t, 0) = h(t, 0) = 0$; *$f(t, x)$ and $h(t, x)$ are quasimonotone in x relative to K,*
(H_6) $D^+(\phi_0, V(t, x)) \le -\nu[G(\phi_0, V(t, x))] + \mu(t)(\phi_0, V(t, x))$,
where $\nu \in \mathcal{K}$, *and* $G, \mu \in C[\Re^+, \Re^+]$.
(H_7) $\mu'(t) < 0$, *and* $\lim_{t\to\infty} \mu(t) = 0$.

If the zero solution of (1.3) *is equi-asymptotically integrally* ϕ_0*-stable, then so is the zero solution of* (1.1).

Proof. By conditions (H_6) and (H_7), there exists a large value of t, say t_1 such that $t_1 > t_0 \ge 0$, for which we have

$$D^+(\phi_0, V(t, x)) \le 0, \qquad \text{for all } t \ge t_1. \tag{2.7}$$

Let the zero solution of (1.3) be equi-asymptotically integrally ϕ_0-stable, then it is integrally ϕ_0-stable, i.e., for a given $\alpha \ge 0$, $t_0 \in J$, there exist $\beta(t_0, \alpha)$, and $\gamma(t_0, \alpha, \epsilon)$, such that

$$(\phi_0, u_0) \le \alpha, \quad \text{and} \quad \int_{t_0}^{t_0+T} p(s)\mathrm{d}s \le \gamma, \quad T > 0, \tag{2.8}$$

implies

$$(\phi_0, r^*) < \beta, \quad t \ge t_0,$$

where r^* is the maximal solution of (1.4). Since condition (H_4) implies condition (H_1), then as in Theorem 2.1, we have

$$(\phi_0, V(t, x^*)) \le (\phi_0, V(t_0, x_0)). \tag{2.9}$$

Thus by Theorem 2.1, the zero solution is equi-integrally ϕ_0-stable.

Now, to prove that $(\phi_0, x^*) \to 0$ as $t \to \infty$, we must prove that $(\phi_0, V(t, x^*)) \to 0$, as $t \to \infty$. Suppose this is false, then $\lim_{t\to\infty} V(t, x) = V^* \neq 0$. Continuing as in [6], using the monotonicity of ν, we have

$$\exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) D^+(\phi_0, V(t, x)) \le -\nu[G(\phi_0, V(t, x))] \exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) + \mu(t)(\phi_0, V(t, x)) \exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right).$$

Thus, if x^* be the maximal solution of (1.2), then

$$D^+(\phi_0, V(t, x^*)) \exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) \le -\nu[G(\phi_0, V(t, x^*))] \exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right). \tag{2.10}$$

By integrating the inequality (2.10), we get

$$(\phi_0, V(t, x)) \exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) \le (\phi_0, V(t_0, x_0)) + \int_{t_0}^{t} -\mu[-G(\phi_0, V)] \exp\left(-\int_{t_0}^{t} \mu(s)\mathrm{d}s\right) \mathrm{d}\mu.$$

Thus as $t \to \infty$, it follows that for some $\phi_0 \in K_0^*$, we have

$$(\phi_0, V(t, x^*)) \to -\infty.$$

This contradicts (H_4). So, $V^* = 0$, and consequently

$$(\phi_0, V(t, x^*)) \to 0 \quad \text{as } t \to \infty. \tag{2.11}$$

Therefore

$$(\phi_0, x^*) \to 0 \quad \text{as } t \to \infty. \tag{2.12}$$

Let $p(t)$ be as in Theorem 2.1, and choose $u_0 = a^{-1}(\|x_0\|)$, then

$$\int_{t_0}^{t_0+T} \|h(s, x^*)\|\mathrm{d}s = \int_{t_0}^{t_0+T} p(s)\mathrm{d}s < \gamma^*, \quad T > 0, \tag{2.13}$$

and

$$(\phi_0, u_0) = (\phi_0, a^{-1}(x_0)) \le a^{-1}(\alpha) < \alpha,$$

i.e.,

$$(\phi_0, u_0) \le \alpha. \tag{2.14}$$

Thus from (2.12)–(2.14), the zero solution of (1.1) is equi-asymptotically integrally ϕ_0-stable. □

Remark. Condition (H_6) is more general than condition (H_1) of Theorem 2.1 in [6] and condition (iii) of Theorem 3.3 in [2].

Theorem 2.3. *Let the zero solution of* (1.1) *be equi-integrally stable and suppose that*

(H_8)

$$\|\Phi(t, s, z)h(s, z)\| \le \lambda(s)\|z\|, \quad \text{and} \quad \int \lambda(s)\|z\|\mathrm{d}s < \infty,$$

where $\Phi(t, t_0, x(t, t_0, x_0))$ *is the fundamental matrix solution of* (1.5), $\lambda \in C[J, \mathfrak{R}^+]$. *Then the zero solution of* (1.1) *is equi-integrally* ϕ_0*-stable.*

Proof. Let $y^*(t, t_0, x_0)$ and $x^*(t, t_0, x_0)$ be the maximal solutions of (1.1) and (1.2) respectively with the same initial value of $y(t_0) = x(t_0) = x_0$. Then by using the nonlinear variation of constants formula, of [1], we have

$$y(t) = x(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, s, y(s))h(s, y)\mathrm{d}s, \tag{2.15}$$

and

$$\|x^*(t)\| \le \|y(t,t_0,x_0)\| + \int_{t_0}^{t} \|\Phi(t,s,y^*(s))h(s,y)\|\mathrm{d}s,$$

where $x^*(t)$ and $y^*(t)$ are the maximal solutions of (1.1) and (1.2) respectively. Now, since the zero solution of (1.1) is equi-integrally stable, for each $\alpha \ge 0$, and $t_0 \in J$, there exists a positive and continuous function in t_0, and $\beta(t_0,\alpha) > 0$ such that for a maximal solution $x(t)$ of (1.2),

$$\|x\| < \beta, \quad t \ge t_0 \quad \text{whenever } \|x_0\| \le \alpha, \text{ and } \int_{t_0}^{t_0+T} h(s,x)\mathrm{d}s \le \alpha. \tag{2.16}$$

Hence by (H_8), we have from (2.15) and (2.16)

$$\|y(t)\| \le \beta + \int_{t_0}^{t} \lambda(s)\|y^*(s)\|\mathrm{d}s.$$

Continuing as in [3] and using Bellman's inequality, we get

$$\|x\| < \beta, \quad t \ge t_0 \quad \text{whenever } \|x_0\| \le \alpha, \text{ and } \int_{t_0}^{t_0+T} h(s,x)\mathrm{d}s \le \alpha.$$

Choose $M\|h(t,x)\| = H(t,x,Lx)$. Thus $\int_{t_0}^{t_0+T} h(s,x,Lx)\mathrm{d}s \le \alpha_1$, and for $\phi_0 \in K_0^*$, we have

$$(\phi_0, x^*) < \beta_1, \quad t \ge t_0 \quad \text{whenever } (\phi_0, x_0) < \alpha_1 \text{ and } \int_{t_0}^{t_0+T} h(s,x^*)\mathrm{d}s \le \alpha_1,$$

where $\beta_1 = \|\phi_0\|\beta$, and $\alpha_1 = \|\phi_0\|\alpha$. Then the zero solution of (1.2) is equi-integrally ϕ_0-stable. □

Theorem 2.4. *Let the hypotheses* (H_1)–(H_3) *be satisfied. If the zero solution of* (1.3) *is uniformly integrally* ϕ_0*-stable, then so is the zero solution of* (1.1).

Proof. Continuing as in the proof of Theorem 2.1, we get

$$(\phi_0, x^*) \le (\phi_0, V(t,x^*)) \le (\phi_0, r^*(t,u_0)). \tag{2.17}$$

From our assumption, there exists $\alpha > 0$, and $\beta(\alpha) > 0$ are independent of t_0 such that

$$(\phi_0, r^*) < \beta, \tag{2.18}$$

provided that

$$(\phi_0, u_0) \le \alpha, \quad \text{and} \quad \int_{t_0}^{t_0+T} p(s)\mathrm{d}s \le \alpha, \tag{2.19}$$

where $r^*(t,t_0,u_0)$ is the maximal solution of (1.4). Thus from (2.1) and (2.18), we get

$$(\phi_0, x^*) \le (\phi_0, V(t,x^*)) \le (\phi_0, r^*(t,u_0)) < \beta$$

choosing

$$u_0 = a(x_0), \quad a(s) \le Ns, \quad a \in \mathcal{K}, \quad N = \frac{\beta}{2\delta\|\phi_0\|}$$

we get

$$(\phi_0, x^*) \le (\phi_0, a(x_0)) \le (\phi_0, Nx_0). \tag{2.20}$$

Similarly as in the proof of Theorem 2.1, choosing $h(t,x) = p(t)$ we obtain

$$\int_{t_o}^{t_o+T} h(s,x)\mathrm{d}s \le \alpha,$$

and then

$$(\phi_0, x^*) < \beta$$

provided that

$$(\phi_0, x_0) \leq \alpha \quad \text{and} \quad \int_{t_0}^{t_0+T} \sup_{\|x\|<\beta} \|h(t,x)\| \mathrm{d}s \leq \alpha.$$

Then the proof completed. □

Remark. It is easy to see that, if the zero solution of (1.1) is uniformly integrally Lipschitz ϕ_0-stable, then it is uniformly integrally stable.

3. More general perturbed systems

Consider the perturbed systems

$$x' = f(t,x) + H(t,x,Lx), \tag{3.1}$$

where $f(t,x)$ be defined as in Section 1, $H \in C[\mathfrak{R}^+ \times S_\rho \times \mathfrak{R}^n, \mathfrak{R}^n]$, L is a continuous operator that map $\mathfrak{R}^n$ to $\mathfrak{R}^n$, and $f_x(t,x) = \frac{\partial f}{\partial x}$ exists and is continuous on $J \times \mathfrak{R}^n$. The stability of system (3.1) was recently discussed from another point of view by the authors (see [3,9–11]).

Let the fundamental matrix solution of

$$y' = f_x(t,0)y$$

be given by

$$\Phi(t,t_0,0) = \frac{\partial x(t,t_0,x_0)}{\partial x_0}.$$

Theorem 3.1. *Suppose that*

(H_9)

$$\|H(t,y,Ly)\| \leq \lambda^*\|y\| + \alpha^*(t)\int_{t_0}^{t} \beta^*(s)\|y\|\mathrm{d}s,$$

where $\lambda^*(t), \alpha^*(t), \beta^*(t) \in C^+(J)$, $J = [t_0, \infty)$.

(H_{10})

$$\int_{t}^{t_0} \alpha^*(s)(\beta^*(\tau)\mathrm{d}\tau)\mathrm{d}s < \infty.$$

If the zero solution of (1.1) *is equi-integrally stable in variation, then the zero solution of* (3.1) *is equi-integrally* ϕ_0*-stable.*

Proof. Let $y(t) = y(t,t_0,x_0)$, and $z(t) = z(t,t_0,x_0)$ be solutions of (1.1) and (3.1) respectively passing through x_0 at $t = t_0$. Continuing as in [3], using the nonlinear variation of constants formula of [1], we get

$$z(t) = y(t) + \int_{t_0}^{t} \Phi(t,s,z(s))H(s,z(s),Lz(s))\mathrm{d}s.$$

Thus

$$\|z(t)\| \leq \|y(t)\| + \int_{t_0}^{t} \|\Phi(t,s,z(s))H(s,z(s),Lz(s))\|\mathrm{d}s, \tag{3.2}$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix solution of (1.5). Now since the zero solution of (1.1) is equi-integral stable in variation, for each $\alpha \geq 0$, $t_0 \in J$, there exists a positive function $\beta(t_0, \alpha)$ which is continuous in t_0, for each $\alpha, t_0 \in J$ such that

$$\|\Phi(t, t_0, x_0)\| < \beta, \quad \text{for } t \geq t_0 \text{ whenever } \|x_0\| \leq \alpha, \int_{t_0}^{t_0+T} \|h(s, x)\| \mathrm{d}s \leq \alpha,$$

where $x(t) = x(t, t_0, x_0)$ is a solution. Thus choosing $H(t, z, L(z)) = Mh(t, z)$, then

$$\int_{t_0}^{t_0+T} \|H(s, z(s), Lz(s))\| \mathrm{d}s \leq \int_{t_0}^{t_0+T} M\|h(s, z(s))\| \mathrm{d}s \leq M\alpha = \alpha_1. \tag{3.3}$$

Now from condition (H_9) and inequalities (3.3) we get

$$\|z\| \leq \beta + \beta \int_{t_0}^{t} \lambda^*(s)\|z\| \mathrm{d}s + \beta \int_{t_0}^{t} \alpha^*(s) \int_{t_0}^{t} \beta(\tau) \mathrm{d}\tau \mathrm{d}s$$

So by Lemma 3 of [7] and using condition (H_{10}), we get

$$\|z\| \leq \beta \left[1 + \int_{t_0}^{t} \lambda^*(s) \exp \int_{t_0}^{t} (\alpha(\tau) + \beta(\tau)) \mathrm{d}\tau\right] \mathrm{d}s < b(\beta),$$

whenever

$$\|x_0\| \leq \alpha_1, \quad \text{and} \quad \int_{t_0}^{t_0+T} \|H(s, z(s), Lz(s))\| \mathrm{d}s \leq \alpha_1.$$

Then the zero solution of (3.1) is equi-integrally stable. For $\phi_0 \in K_0^*$, $t_0 \in J$, and the maximal solution $z^*(t)$ of (3.1), we get

$$(\phi_0, z^*(t)) < \beta_2,$$

whenever

$$(\phi_0, x_0) \leq \alpha_2 \quad \text{and} \quad \int_{t_0}^{t_0+T} \|H(s, z(s), Lz(s))\| \leq \alpha_2,$$

where $\beta_2 = \|\phi_0\| b(\beta)$ and $\alpha_2 = \|\phi_0\|\alpha_1$. Then the zero solution of (3.1) is equi-integrally ϕ_0-stable. □

Remark. In the special case for which the operator L is defined by

$$Lx(t) = \int_{0}^{t} g(t, s, x(s)) \mathrm{d}s \quad 0 \leq t_0 \leq s \leq t < \infty.$$

The perturbed system (3.1) includes the integrodifferential equation

$$x' = f(t, x) + \int_{t_0}^{t} g(t, s, x(s)) \mathrm{d}s, \tag{3.4}$$

which has recently received great interest from many authors (see [4,7,8]).

Remark. We think that this work can be done for other general notions, for example integral Lipschitz ϕ_0-stability and integral Lipschitz stability.

4. Example

Consider the system

$$x' = x^2 \mathrm{e}^{-t}, \quad x(t_0) = x_0, \tag{4.1}$$

and the perturbed system

$$x' = x^2 \mathrm{e}^{-t} + h(t, x), \quad x(t_0) = x_0. \tag{4.2}$$

In fact, it is clear that the fundamental matrix solution of the corresponding variation system is

$$\Phi(t, t_0, x_0) = \frac{1}{[1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0})]^2},$$

and

$$x(t, t_0, x_0) = \frac{x_0}{1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0})},$$

is a solution of (4.1). Let

$$V(t, x) = x^2, \tag{4.3}$$

then

$$D^+ V(t, x) = 2x\,\Phi(t, t_0, x_0)h(t, x).$$

Now, if for simplicity we choose $h = \frac{x}{2}$, then for the maximal solution $x^*(t, t_0, x_0)$ of (4.2), we have

$$\|x^*(t, t_0, x_0)\|^2 \leq \frac{\|x_0\|^2}{[1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0}) + (t - t_0/2)]^2}.$$

Also we note that, condition (H_2) is satisfied, i.e. V is Lipschitzian in x, and we have for $\phi_0 \in K_0^*$

$$(\phi_0, x) \leq (\phi_0, V) \leq (\phi_0, x^2).$$

Furthermore

$$\begin{aligned} D^+(\phi_0, V) &= D^+(\phi_0, x^2) \\ &\leq (\phi_0, 2x\,\Phi h) \\ &= \left(\phi_0, \frac{x^2}{[1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0})]^2}\right) \\ &\leq \|\phi_0\| \left\| \frac{x^2}{[1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0})]} \right\| \\ &= \|\phi_0\| \frac{\|x^2\|}{\|[1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0})]\|}, \end{aligned}$$

where in the system $u' = g(t, u)$, the function g is

$$g = \frac{u^2}{1 + x_0(\mathrm{e}^{-t} - \mathrm{e}^{-t_0})}.$$

Hence all the conditions of Theorem 2.1 are satisfied i.e., the zero solution of (4.1) is equi-integrally ϕ_0-stable. Moreover, for the choice

$$g = -\gamma[G(\phi_0, x^2)] + \mu[(\phi_0, x^2)],$$

where

$$\mu(t) = \frac{\mathrm{e}^{-t}}{\|\phi_0\|}, \qquad G = \frac{\|t\|}{\|\phi_0\|}, \quad \text{and} \quad \gamma = \frac{\|t^2\|}{\|\phi_0\|},$$

for $\phi_0 \in K_0^*$, $t \geq t_0$, we see that all the conditions of Theorem 2.2 are satisfied, and we can directly get

$$(\phi_0, x^*) \to, \quad \text{as } t \to \infty,$$

i.e., the zero solution of (4.1) is equi-asymptotically integrally ϕ_0-stable.

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References

[1] V.M. Alekseev, An estimate for perturbations of the solutions of ordinary differential equations, Vestnik Moskov-Univ. Ser. 1 Math. Mekh. 2 (1964) 28–36.
[2] E.P. Akpan, O. Akinyele, On the ϕ_0-stability of comparison differential systems, J. Math. Anal. Appl. 164 (1992) 307–324.
[3] F. Dannan, S. Elaydi, Lipschitz stability of nonlinear systems of differential equations, J. Math. Anal. Appl. 113 (2) (1986) 562–577.
[4] F. Dannan, S. Elaydi, Lipschitz stability of nonlinear systems of differential equations II, J. Math. Anal. Appl. 143 (2) (1989) 517–529.
[5] M.M. El-Sheikh, A.A. Soliman, On Lipschitz stability for nonlinear systems of ordinary differential equations, J. Diff. Eqs. Dynam. Syst. 3 (3) (1995) 235–250.
[6] M.M. El-Sheikh, A.A. Soliman, ϕ_0- stability criteria of nonlinear systems of differential equations, Pan. Amer. Math. J 5 (3) (1995) 17–30.
[7] V. Lakshmikantham, S. Leela, Differential and Integral Inequalities, vol. 1, Academic press, New York, 1969.
[8] V. Lakshmikantham, S. Leela, Cone valued Liapunov functions, Math. Non Linear Anal. 1 (1977) 215–222.
[9] B.G. Pachpatte, Stability and asymptotic behavior of perturbed nonlinear systems, J. Differential Equations 16 (1976) 14–25.
[10] B.G. Pachpatte, Perturbations of nonlinear systems of differential equations, J. Math. Anal. Appl. 51 (1975) 550–556.
[11] S. Rajalaksmy, S. Sivasudaram, Vector Liapunov functions and the technique in perturbations theory, J. Math. Anal. Appl. 164 (1992) 660–670.
[12] S. Zhang, Comparison theorems on boundedness, Funk, Ekva 31 (2) (1988) 179–196.